Unconventional Logic for Massively Parallel Reasoning

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Abstract. **In this paper, unconventional logic for massive-parallel reasoning is introduced and two examples of simulating natural processes by means of this logic are regarded: chemical (Belousov-Zhabotinsky reaction) and biological (dynamics of plasmodium of *Physarum polycephalum*). In unconventional logic well-formed formulas are defined as streams and there are possible so-called wave sets of formulas (where all members are non-well-founded). This way of simulating is much simpler in applications of unconventional computing than sequential thinking.**

***Keywords*:** **proof-theoretic cellular automata; Belousov-Zhabotinsky reaction; Physarum polycephalum, wave sets, streams.**

I. Introduction

THE elementary way of simulating massive parallelism in the nature is presented by cellular automata. Stephen Wolfram claims in his famous book *A New Kind of Science* [7] that all basic physical and biological processes may be regarded as computer's programs, namely as cellular-automatic simulations. In this paper we discuss the possibility to establish massive-parallel reasoning for simulating natural processes and to construct an appropriate unconventional logic, where all formulas are built as streams (i.e. they are non-well-founded according to set theory).

Recall that any cellular automaton consists of cells belonging to the set $\mathbf{Z}^d$, thereby each of cell takes its value in S, a finite or infinite set of elements called the states of an automaton. Usually, cells are considered as unchangeable, but their states change permanently. This dynamics depends on local transition rule δ: $S^{n+1} \to S$ that transforms states of cells taking into account states of n neighbor cells. The ordered set of n elements, N, is said to be a neighborhood. Each step of dynamics is fixed by discrete time $t = 0, 1, 2, \ldots$

At the moment t, the configuration of the whole system (or the global state) is given by the mapping x^t from $\mathbf{Z}^d$ into S, and the evolution is the sequence $x^0, x^1, x^2, \ldots$ defined as follows:

$$x^{t+1}(z) = \delta(x^t(z), x^t(z+\alpha_1), \ldots, x^t(z+\alpha_n)),$$

where $\langle \alpha_1, \ldots, \alpha_n \rangle \in N$.

Here x^0 is the initial configuration, and it fully determines the future behavior of the automaton.

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For any logical language L we can construct a *proof-theoretic cellular automaton* (instead of conventional deductive systems) simulating massive-parallel proofs. In a proof-theoretic cellular automaton the set S of states of an automaton A is collected from well-formed formulas of a language L and the role of local transition function, δ, is played by the inference rule of a language L. The initial configuration of A is the set of all premises (not axioms) and it fully determines the future behavior of the automaton. We assume that δ is an inference rule, i.e. a mapping from the set of premises (their number cannot exceed $n = |N|$) to a conclusion. For any $z \in \mathbf{Z}^d$ the sequence $x^0(z)$, $x^1(z)$, $x^2(z)$, …, $x^t(z)$, … is called a *derivation trace from a state* $x^0(z)$. If there exists t such that $x^t(z) = x^l(z)$ for all $l > t$, then a derivation trace is *finite*. It is *circular/cyclic* if there exists l such that $x^t(z) = x^{t+l}(z)$ for all t. In case all derivation traces of a proof-theoretic cellular automaton A are circular, this automaton A is said to be *reversible.*

In the next two sections we consider a massive-parallel simulation of two natural processes: Belousov-Zhabotinsky reaction and dynamics of plasmodium of *Physarum polycephalum.*

II. Massive-Parallel Proofs for Belousov-Zhabotinsky Reaction

The mechanism of the Belousov-Zhabotinsky reaction (namely cerium(III) ↔ cerium(IV) catalyzed reaction) is very complicated: its recent model contains 80 elementary steps and 26 variable species concentrations. Let us consider a simplification of Belousov-Zhabotinsky reaction assuming that the set of states consists just of the following reactants: Ce^{3+}, $HBrO_2$, BrO^-_3, H^+, Ce^{4+}, H_2O, $BrCH(COOH)_2$, Br^-, $HCOOH$, CO_2, $HOBr$, Br_2, $CH_2(COOH)_2$ which interact according to basic reactions of Figure 1. In this reaction we observe sudden oscillations in color from yellow to colorless, allowing the oscillations to be observed visually. In spatially non-homogeneous systems (such as a simple Petri dish), the oscillations propagate as spiral wave fronts. The oscillations last about one minute and are repeated over a long period of time. The color changes are caused by alternating oxidation-reductions in which cerium changes its oxidation state from cerium(III) to cerium(IV) and vice versa: $Ce^{3+} \to Ce^{4+} \to Ce^{3+} \to \ldots$.

When Br^- has been significantly lowered, the reaction causes an exponential increase in bromous acid ($HBrO_2$)

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and the oxidized form of the metal ion catalyst and indicator, cerium(IV). Bromous acid is subsequently converted to bromate (BrO^{-}_3) and $HOBr$. Meanwhile, the next step reduces the cerium(IV) to cerium(III) and simultaneously increases bromide (Br^{-}) concentration. Once the bromide concentration is high enough, it reacts with bromate (BrO^{-}_3) and $HOBr$ to form Br_2, further Br_2 reacts with $CH_2(COOH)_2$ to form $BrCH(COOH)_2$ and the process begins again. Thus, parallel processes have several cycles which are performed synchronously.

The proof-theoretic simulation of Belousov-Zhabotinsky reaction can be defined as follows (it is a generalization of chemical machine [5] or reaction-diffusion computing [3], [1]):

Definition 1. Let $\{Ce^{3+}, HBrO_2, BrO^{-}_3, H^{+}, Ce^{4+}, H_2O, BrCH(COOH)_2, Br^{-}, HCOOH, CO_2, HOBr, Br_2, CH_2(COOH)_2\}$ be the set of states. The two basic operations $\vee$ (disjunction) and & (fusion) are defined by inference rules in Figure 1, which describe general properties of transitions. Then the proof-theoretic simulation of Belousov-Zhabotinsky reaction is a concurrent automaton.

In this system, transitions between states are identified with derivations determined by initial values of states. Here we obtain a massive-parallel process of derivation, too. Each step of derivation means a transition. As a result, the circular trace of state Ce^{3+} (resp. Ce^{4+}) has a meaning of circular proof, where the state Ce^{3+} (resp. Ce^{4+}) is unfolded infinitely often among premises and at the same time among derivable expressions.

$$\frac{Ce^{3+} \,\&\, HBrO_2 \,\&\, BrO_3^{-} \,\&\, H^{+}}{Ce^{4+} \vee HBrO_2 \vee H_2O}$$

$$\frac{Ce^{4+} \,\&\, BrCH(COOH)_2 \,\&\, H_2O}{Br^{-} \vee Ce^{3+} \vee HCOOH \vee CO_2 \vee H^{+}}$$

$$\frac{HBrO_2}{HOBr \vee BrO_3^{-} \vee H^{+}} \qquad \frac{BrO_3^{-} \,\&\, Br^{-} \,\&\, H^{+}}{HOBr \vee HBrO_2}$$

$$\frac{HBrO_2 \,\&\, Br^{-} \,\&\, H^{+}}{HOBr} \qquad \frac{HOBr \,\&\, Br^{-} \,\&\, H^{+}}{Br_2 \vee H_2O}$$

$$\frac{Br_2 \,\&\, CH_2(COOH)_2}{Br^{-} \vee H^{+} \vee BrCH(COOH)_2}$$

Fig. 1. The basic reactions of Belousov-Zhabotinsky reaction, where the sign $\vee$ is a disjunction, which means that at least one disjunctive state takes place (in concurrent inference rules we can use any member of disjunction as premise for further deducing), and the sign & is a fusion that means in concurrent automata the fusion of active cells if they are neighbors.

III. Massive-Parallel Proofs for Dynamics of Plasmodium of Physarum Polycephalum

The dynamics of plasmodium of *Physarum polycephalum* could be regarded as another simple example of the natural proof-theoretic automata. The point is that when the plasmodium is cultivated on a nutrient-rich substrate (agar gel containing crushed oat flakes) it exhibits uniform circular growth similar to the excitation waves in the excitable Belousov-Zhabotinsky medium (see Figure 2). If the growth substrate lacks nutrients, e.g. the plasmodium is cultivated on a non-nutrient and repellent containing gel, a wet filter paper or even glass surface localizations emerge and branching patterns become clearly visible (see Figure 2).

The plasmodium continues its spreading, reconfiguration and development as long as there are enough nutrients. When the supply of nutrients is over, the plasmodium either switches to fructification state (if level of illumination is high enough), when sporangia are produced, or forms sclerotium (encapsulates itself in hard membrane), if in darkness.

The pseudopodium propagates in a manner analogous to the formation of wave-fragments in sub-excitable Belousov-Zhabotinsky systems. Starting in the initial conditions the plasmodium exhibits foraging behavior, searching for sources of nutrients. When such sources are located and taken over, the plasmodium forms characteristic veins of protoplasm, which contracts periodically. Belousov-Zhabotinsky reaction and plasmodium are light-sensitive, which gives us the means to program them. *Physarum* exhibits articulated negative phototaxis, Belousov-Zhabotinsky reaction is inhibited by light. Therefore by using masks of illumination one can control the dynamics of localizations in these media.

Experiments with *Physarum polycephalum* were carried out by Prof. Adamatzky [2] as follows. The plasmodia of *Physarum polycephalum* were cultured on wet paper towels, fed with oat flakes, and moistened regularly. He subcultured the plasmodium every 5 – 7 days.

Experiments were performed in standard Petri dishes, 9 cm in diameter. Depending on particular experiments he used 2% agar gel or moisten filter paper, nutrient-poor substrates, and 2% oatmeal agar, nutrient-rich substrate (Sigma-Aldrich). All experiments were conducted in a room with diffusive light of 3 – 5 cd/m, 22°C temperature. In each experiment an oat flake colonized by the plasmodium was placed on a substrate in a Petri dish, and few intact oat flakes distributed on the substrate. The intact oat flakes acted as source of nutrients, attractants for the plasmodium. Petri dishes with plasmodium were scanned on a standard HP scanner. The only editing done to scanned images is color enhancement: increase of saturation and contrast.

Results of experiments may be described in terms of proof-theoretic spatial automata. Let us assume that its set of states consists of the entities from the following sets.

1. The set of *growing pseudopodia*, $\{P_1, P_2, ...\}$, localized in *active zones*. On a nutrient-rich substrate plasmodium propagates as a typical circular, target wave, while on the nutrient-poor substrates localized wave-fragments are formed.
2. The set of *attractants* $\{A_1, A_2, \dots\}$, they are sources of nutrients, on which the plasmodium feeds. It is still subject of discussion how exactly plasmodium feels

presence of attracts, indeed diffusion of some kind is involved. Based on previous experiments by Prof. Adamatzky we can assume that if the whole experimental area is about 8 – 10cm in diameter then the plasmodium can locate and colonize nearby sources of nutrients.

3. The set of *repellents* $\{R_1, R_2, \ldots\}$. Plasmodium of *Physarum* avoids light. Thus, domains of high illumination are repellents such that each repellent R is characterized by its position and intensity of illumination, or force of repelling.
4. The set of *protoplasmic tubes* $\{C_1, C_2, \ldots\}$. Typically plasmodium spans sources of nutrients with protoplasmic tubes/veins. The plasmodium builds a planar graph, where nodes are sources of nutrients, e.g. oat flakes, and edges are protoplasmic tubes.

Hence, the set of states in the proof-theoretic spatial automaton for dynamics of Plasmodium of *Physarum polycephalum* is equal to $\{P_1, P_2, ...\} \cup \{A_1, A_2, \ldots\} \cup \{R_1, R_2, \ldots\} \cup \{C_1, C_2, \ldots\}$.

The proof-theoretic simulation of *Physarum polycephalum* is defined as follows:

Definition 2. Consider a propositional language L with the only binary operation $\vee$, it is built in the standard way over the set of variables $S = \{P_1, P_2, ..., A_1, A_2, \ldots, R_1, R_2, \ldots, C_1, C_2, \ldots\}$. Let S be the set of states of proof-theoretic spatial automaton A. Assume that 0 is the empty cell. The inference rule of the automaton is as follows:

$$x^{t+1}(z) = \begin{cases} X \vee P_j \ if \ x^t(z) = X \vee C_n \vee A_i \\ X \vee P_j \ if \ x^t(z) = X \vee P_j \vee A_i \\ (X \vee P_i) \vee (X \vee P_j) \ if \ x^t(z) = X \vee P_i \vee P_j \\ (X \vee C_i) \vee (X \vee C_j) \ if \ x^t(z) = X \vee C_i \vee C_j \\ 0 \vee C_i \ if \ x^t(z) = 0 \ and \ premises \ R_i \notin (z+N), \\ \qquad premises \ C_j, A_i \in (z+N); \\ 0 \vee C_i \ if \ x^t(z) = 0 \ and \ premises \ R_i \notin (z+N), \\ \qquad premises \ P_j, A_i \in (z+N); \\ x^t(z), otherwise. \end{cases}$$

Then A simulates the dynamics of Plasmodium of *Physarum polycephalum*.

Definition 3. Let $p, s_i, s_{i+1} \in \{P_1, P_2, \ldots, A_1, A_2, \ldots, R_1, R_2, \ldots, C_1, C_2, \ldots\}$. A state p is called a premise for deducing s_{i+1} from s_i by inference rules of definition 2 iff either p is s_i or in a neighbor cell we find out an expression of the form $p \vee X$, where X is a propositional metavariable, i.e. it runs over either the empty set or the set of states closed under the operation $\vee$. Thus, we assume that each premise should occur in a separate cell. This means that if we find out an expression $p_i \vee p_j \vee X$ in a neighbor cell and both p_i and p_j are needed for deducing, whereas p_i, p_j do not occur in other neighbor cells, then p_i, p_j could not be considered as premises.

An example of the evolution of a proof-theoretic spatial automaton with the neighborhood of 8 members in the 2-dimensional space for *Physarum polycephalum* is pictured in Figure 3.

IV. Unconventional Logic of Massive-Parallel Reasoning

Thus, proof-theoretic cellular automata can simulate many chemical and biological processes. Does it mean that on media of some natural processes (chemical or biological phenomena) we can project an unconventional computer (with logical circuits on Belousov-Zhabotinsky or *Physarum polycephalum* medium)? No, it does not, if we appeal just to proof-theoretic cellular automata, because they logically only describe natural processes, but they do not show us the way how to use the nature as an unconventional computer. In this section we try to sketch an unconventional logic, within which we can combine different proof-theoretic cellular automata. The possibility of these combinations means already that using this unconventional logic we can logically project an abstract unconventional computer on any natural media (like Belousov-Zhabotinsky or *Physarum polycephalum*).

Let us remind that each word in conventional logical language is defined by induction and possesses the form $\alpha\beta_1, \ldots, \beta_n$, where α is a sign (conjunction, disjunction, implication, negation, etc.), $\beta_1, \ldots, \beta_n$ are words. If $n = 0$, α is a propositional variable. Otherwise it is one of logical connectives.

Proposition 1. Each word can be written as $\alpha\beta_1, \ldots, \beta_n$ in a unique way.

Proof. The first symbol in a word is α. It is a sign. If $\alpha\beta_1, \ldots, \beta_n$ and $\alpha\beta'_1, \ldots, \beta'_n$ are words that are equal, then by induction we have that β_1 is equal to β'_1, β_2 is equal to β'_2, etc. Q.E.D.

In massively parallel reasoning we prefer to deal with the whole and ignore details. This means that a composition in logical syntax may be non-linear. Such non-linear composite words will be considered by us as *wave set data* or *wave sets*. For example, the maximum of all decision paths in reflexive games can be examined as wave set. Let us try to define wave sets mathematically.

In set theory, any set A is defined by its members $a, b, c, \ldots$, i.e. $A = \{a, b, c, \ldots\}$. According to a set-theoretic axiom, the so-called foundation axiom, we can build up a hierarchy of sets:

Axiom of Foundation. If A is a non-empty family of sets, then there exists a set X such that $X \in A$ and $X \cap A = \varnothing$.

From this axiom it follows that there is no X such that $X \in X$. Let us deny this axiom and postulate the existence of the following entity:

Definition 4. Let A be a non-empty set. This set is called *wave set* if and only if for any $X \in A$ we have $X \cap A \neq \varnothing$.

Let us consider an example if $A = \{a, b\}$, where $a = \{a, \{b\}\}$ and $b = \{b, \{a\}\}$. We can readily see that A is a wave set. In order to differ wave sets from conventional sets let us denote wave sets of the form $A = \{a, b\}$ by $A = (\}a, b\{)$.

Formulas may be presented as wave sets, too. For instance, let $\Phi = (\} \psi, \varphi \{)$ be a composite formula consisting of simpler formulas ψ and φ, where $\varphi = \varphi \Rightarrow \psi$ and $\psi = \psi \wedge \varphi$. Then the formula Φ is a wave set. We cannot build up such formulas in conventional symbolic logic. The matter is that in conventional logical syntax all formulas are composed as inductive sets and in turn all inductive sets are well-founded (i.e. satisfy the foundation axiom).

Massive-parallel reasoning may be formalized as a non-linear word of the form $\Phi = (\}\alpha\beta_1, \beta_2, \ldots, \ldots, \alpha\beta'_1, \beta'_2, \ldots\{)$ consisting at least of two words $\alpha\beta_1, \beta_2, \ldots$ and $\alpha\beta'_1, \beta'_2, \ldots$, which are infinite and we cannot learn whether Φ is either $\alpha\beta_1, \beta_2, \ldots$ or $\alpha\beta'_1, \beta'_2, \ldots$ For instance, if $\varphi = \varphi \Rightarrow \psi$ is a stream, then we have an infinite formula $(((\ldots) \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$ and if $\psi = \psi \wedge \varphi$ is a stream, then there is an infinite formula $(((\ldots) \wedge \varphi) \wedge \varphi) \wedge \varphi$. The composite formula $\Phi = (\}\psi, \varphi\{)$ cannot be presented by linear words even if they are infinite. It is non-linear in principle. We do not know it is either an infinite implication or an infinite conjunction. Nevertheless, by composition Φ is both implication and conjunction.

Wave sets are the way to see the whole. The wave set includes as many versions of behavior of the whole system as possible. For example, assume we have several cellular automata for *Physarum polycephalum* with the same transition rules. They are distinguished by initial states. The evolution of each cell is a stream. The different automata assume different streams for some cells. The wave set for all automata A for *Physarum polycephalum* contain each possible stream of each cell.

Definition 5. The wave set of cellular automata does not depend on initial states; it contains all possible evolutions of each cell in accordance with one transition rule.

Let $\{\vee, \wedge, \Rightarrow, \neg\}$ be the set of standard logical operations (disjunction, conjunction, implication, negation, respectively) and $\{p, q, r, \ldots\}$ be the set of all propositional variables. Then formulas are defined as follows: if $\varphi \in \{p, q, r, \ldots\}$ and ψ is a formula, then $\varphi \circ \psi$, where $\circ \in \{\vee, \wedge, \Rightarrow, \neg\}$, is a formula. According to this definition, we construct streams of the form $\varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_n \circ \ldots$, where $\circ \in \{\vee, \wedge, \Rightarrow, \neg\}$ and $\varphi_i \in \{p, q, r, \ldots\}$ for i=0,1,… The class of all streams is a maximal set of formulas closed under logical operations by definition above. That is, this set is defined non-inductively, although it contains all conventional propositional formulas that are defined inductively.

In order to construct the language of massive-parallel reasoning for a formal theory T with a model M, first we should build the diagram language L_M from a formal language L of T by adding constant symbols c_m for all elements m from M, the support set of model M. The set D(M) of all atomic formulas and their negations in language L_M, true in M, is called the diagram of system M. The elementary diagram of system M is said to be theory Th(M), i.e. the set of all propositions well-formed in L_M which are true in M. A start-up theory for building massive-parallel reasoning is presented by an elementary diagram. The properties of any elementary diagram are as follows: (i) Th(M) is consistent; (ii) if $m, n \in M$ and $m \neq n$, then $c_m \neq c_n$ in Th(M).

Definition 6. A logical language for massive-parallel reasoning L_M consists of the following symbols: (i) a non-empty set of constant symbols c_m, c_n, … ; (ii) a non-empty set of function symbols of arity n: $f_0^n, f_1^n, f_2^n, \ldots$; (iii) a non-empty set of predicate symbols of arity n: P_0^n, P_1^n, $P_2^n, \ldots$; (iv) the truth constants 0, 1; (v) propositional connectives of arity n_j: $\mathsf{W}_0^{n_0}$, $\mathsf{W}_1^{n_1}, \ldots, \mathsf{W}_r^{n_r}$, which are built by superposition of $\vee, \wedge, \Rightarrow, \neg$; (vi) first-order quantifiers $\forall, \exists$; (vii) auxiliary symbols (,).

Terms and well-formed atomic formulas of L_M are defined non-inductively:

Definition 7. Terms are defined as follows: (i) if $f_i^n(t_1, \ldots, t_n)$ is a term and $t_1 \in \{c_m, c_n, \ldots\}$, then $t_1, \ldots, t_n$ are terms.

Well-formed formulas of L_M have a non-inductive definition, too:

Definition 8. Formulas are defined as follows: (i) if $t_1, \ldots, t_n$ are terms, then $P_i^n(t_1, \ldots, t_n)$ is an atomic formula; (ii) if $\mathsf{W}_i^{n_i}(x_1, \ldots, x_n)$ is a formula and x_1 is atomic, then $x_1, \ldots, x_n$ are formulas.

The above definition assumes that the formula is a syntactic object which does not satisfy the set-theoretic axiom of foundation. It means that the given syntactic objects can already be of an infinite length and comprise cycles. For example, cyclic expressions $(\varphi_1 \Rightarrow (\varphi_2 \Rightarrow (\varphi_1 \Rightarrow (\varphi_2 \Rightarrow (\ldots)))))$ or $(\varphi_1 \wedge (\varphi_1 \wedge (\varphi_1 \wedge (\varphi_1 \wedge (\ldots)))))$ are well-formed formulas. These expressions can be defined by recursion, too. The formula of infinite length $(\varphi_1 \Rightarrow (\varphi_2 \Rightarrow (\varphi_1 \Rightarrow (\varphi_2 \Rightarrow (\ldots)))))$ is equivalent to the cyclic definition $\varphi = (\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi))$. The formula of infinite length $(\varphi_1 \wedge (\varphi_1 \wedge (\varphi_1 \wedge (\varphi_1 \wedge (\ldots)))))$ is equivalent to the cyclic definition $\varphi = (\varphi_1 \wedge \varphi)$.

Now we can define wave sets in L_M. First, terms may act as wave sets. Then we can construct formulas as wave sets. So, the following expression

$$P\begin{pmatrix} \mathrm{Q}x_\alpha & \mathrm{Q}x_\beta \\ \mathrm{Q}y_\gamma & \mathrm{Q}y_\delta \end{pmatrix} \quad (1)$$

may be considered as well-formed, where Q is a stream (infinite logical operation, e.g. infinite disjunction or infinite conjunction), x_ξ, x_η, y_σ, y_τ are atomic formulas or their negations for any $\xi < \alpha$, $\eta < \beta$, $\sigma < \gamma$, $\eta < \delta$ such that x_ξ, x_η, y_σ, y_τ are used in streams of (1). The formula (1) contains at least two streams, which are mutually defined. Then (1) is called a wave set of L_M. It is denoted by **P**.

A partly truth valuation of a well-formed formula **P** is a function $f_{\mathbf{P}}$ defined on one variable of $\{x_\xi, x_\eta, y_\sigma, y_\tau, \ldots\}$ for any $\xi < \alpha$, $\eta < \beta$, $\sigma < \gamma$, $\eta < \delta$ and has meanings 1 (‘true’) or 0 (‘false’):

$$f_{\mathbf{P}}(x_\xi) = \begin{cases} 1, & \text{if } x_\xi \text{ occurs in } \mathbf{P}; \\ 0, & \text{otherwise.} \end{cases}$$

If $f_{\mathbf{P}}(x_\xi) = 1$, we say that **P** is true on x_ξ.

A total truth valuation of a well-formed formula **P** is a function $F_{\mathbf{P}}$ defined on all atomic formulas built up in the language. It assigns 1 or 0 for all atomic formulas. The total valuation on an atomic formula of the whole formula is equal to an appropriate partly truth valuation of **P**. According to a total truth valuation, **P** is valid if and only if it contains all atomic formulas:

$$F_{\mathbf{P}}(\{x_\xi, x_\eta, y_\sigma, y_\tau, \ldots\}) = \begin{cases} 1, & \text{if everyone of } \{x_\xi, x_\eta, \ldots\} \text{ occurs in } \mathbf{P}; \\ 0, & \text{otherwise.} \end{cases}$$

All total truth valuations $F_{\mathbf{P}}$, $F_{\mathbf{P}'}$, $F_{\mathbf{P}''}$, … are ordered as follows: $F_{\mathbf{P}^k} \le F_{\mathbf{P}^{k+1}}$ if and only if $F_{\mathbf{P}^k} \vee F_{\mathbf{P}^{k+1}} = F_{\mathbf{P}^{k+1}}$. According to this definition, a truth valuation of valid formula is greatest. By this order, any wave sets $\mathbf{P}$, $\mathbf{P}'$, $\mathbf{P}''$, $\mathbf{P}'''$, … such that $F_{\mathbf{P}} \le F_{\mathbf{P}'} \le F_{\mathbf{P}''} \le F_{\mathbf{P}'''} \le \ldots$ has its limit presented by a valid formula.

Now let us try to sketch model theory of unconventional propositional language L_M containing wave sets. In the beginning we should define meanings of unconventional logical connectives ∨, ∧, ⇒, ¬ defined on streams of propositional variables or atomic formulas. They cannot be verified on truth tables, because they assume infinite and changeable streams of truth values.

Let us take a cellular automaton A = <$\mathbf{Z}^d$, S, N, δ>, where

- $d \in N$ is a number of dimensions and the members of $\mathbf{Z}^d$ are referred as cells,
- S is a set of elements called the states of an automaton A, the members of $\mathbf{Z}^d$ take their values in S, the set S is collected from two truth values 1 and 0 for well-formed formulas of the language L_M.
- $N \subset \mathbf{Z}^d \setminus \{0\}^d$ is a finite ordered set of n elements, N is said to be a neighborhood,
- $\delta : S^{n+1} \to S$ that is $\delta \in \{\wedge, \vee, \Rightarrow, \neg\}$ is the truth valuation for each logical operation of the language L_M, it plays the role of local transition function of the automaton A.

Definition 9. A truth valuation of infinite conjunction, i.e. of a stream $\varphi \wedge \varphi \wedge \varphi \wedge \ldots$, is a transition rule of the automaton A, where S = {0, 1}, and it is formulated as follows:

$$x^{t+1}(z) = \begin{cases} 1, & \textit{if } x^t(z) = 1 \textit{ and } 0 \notin (z+N); \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 10. A truth valuation of infinite disjunction, i.e. of a stream $\varphi \vee \varphi \vee \varphi \vee \ldots$, is a transition rule of the automaton A, where S = {0, 1}, and it is formulated as follows:

$$x^{t+1}(z) = \begin{cases} 1, & \textit{if } x^t(z) = 1 \textit{ or } 1 \in (z+N); \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 11. A truth valuation of infinite implication $\varphi \Rightarrow (\varphi \Rightarrow (\varphi \Rightarrow \ldots))$ is a transition rule of the automaton A, where S = {0, 1}, and it is formulated as follows:

$$x^{t+1}(z) = \begin{cases} 0, & \textit{if } x^t(z) = 0 \textit{ and } 0 \notin (z+N); \\ 1, & \textit{otherwise.} \end{cases}$$

Definition 12. A truth valuation of infinite negation $\ldots\neg(\neg(\neg\varphi))$ is a transition rule of the automaton A, where S = {0, 1}, and it is formulated as follows:

$$x^{t+1}(z) = \begin{cases} 0, & \textit{if } x^t(z) = 1; \\ 1, & \textit{otherwise.} \end{cases}$$

Algorithm 1 (for truth valuation of any propositional stream) Let φ be a propositional stream. Its initial truth value is an initial configuration of a cellular automaton A with the set of states {0, 1}. Then we start the evaluation of φ with the very first connective k_1 that occurs in φ. By using one of definitions 9 – 12 that corresponds to k_1, we transform the initial configuration of A at time t=0 to a configuration at time t=1. Further, we move to evaluate the second connective k_2 that occurs in φ. By using one of definitions 9 – 12 that corresponds to k_2, we transform the configuration of A at time t=1 to a configuration at time t=2. Then we go to evaluate the third connective k_3 that occurs in φ. By using one of definitions 9 – 12 that corresponds to k_3, we transform the configuration of A at time t=2 to a configuration at time t=3, etc.

In order to provide an example, let us evaluate the propositional stream $p \Rightarrow (q \wedge r \wedge \ldots)$ in a cellular automaton A with the Moor neighborhood in the 2-dimensional space:

(I) Initial configuration, t=0

1	0	0
0	1	1
0	1	0

We begin our evaluation with the connective ⇒ as the first. Thereby we are using definition 11:

(II) t=1

1	1	1
1	1	1
1	1	0

Then we move to the second connective ∧ and by definition 9 we obtain the following data over (II):

(III) t=2

1	1	1
1	0	0
1	0	0

(IV) t=3

0	0	0
0	0	0
0	0	0

Step (IV) is for further modification in accordance with next connectives of the stream $p \Rightarrow (q \wedge r \wedge \ldots)$.

For wave sets of L_M, truth evaluations are defined as wave sets of appropriate cellular automata:

Definition 13. Let $P\begin{pmatrix} Qx_\alpha & Qx_\beta \\ Qy_\gamma & Qy_\delta \end{pmatrix}$ be a wave set containing just the four (or n) propositional streams: Qx_α, Qx_β, Qy_γ, Qy_δ. A truth valuation of

$$P\begin{pmatrix} Qx_\alpha & Qx_\beta \\ Qy_\gamma & Qy_\delta \end{pmatrix}$$

is a wave set of the four (or n) cellular automata which are built for streams Qx_α, Qx_β, Qy_γ, Qy_δ in the accordance with Algorithm 1.

If the set of terms of L_M is not empty, it is more convenient to appeal to coalgebraic systems, e.g. labeled transition systems, instead of cellular automata. Recall that labeled transition systems are defined as systems $\langle S, \rightarrow_S, A \rangle$, consisting of a set S of states, a transition relation $\rightarrow_S \subseteq S \times A \times S$, and a set A of labels. As always, $s \rightarrow_S^a s'$ is used to denote $\langle s, a, s' \rangle \in \rightarrow_S$.

Cellular automata $\mathsf{A} = \langle \mathbf{Z}^d, S, N, \delta \rangle$ may be regarded as a kind of labeled transition systems of the form $\langle S, \rightarrow_S, \{\delta_i\}_i \rangle$, where S is a set of states, $\rightarrow_S \subseteq S \times \{\delta_i\}_i \times S$, and $\{\delta_i\}_i$ is a set of labels for the local transition function δ of an automaton A. The members of $\{\delta_i\}_i$ is obtained as follows. We have i different combinations of neighbor states. This number depends on the number of neighborhood $n = |N|$ and the number of states $k = |S|$, namely $i = kn$. Hence, $|\{\delta_i\}_i| = i$. Each combination of i has its own label $\delta_{1 \leq j \leq i}$.

Further let us define

$$B(X) = \mathsf{P}(A \times X) = \{V : V \subseteq A \times X\} \quad \text{for any set } X.$$

Then a labeled transition system $\langle S, \rightarrow_S, A \rangle$ can be represented as a B-coalgebra $\langle S, \alpha_S \rangle$ by defining

$$\alpha_S : S \rightarrow B(S); s \rightarrow \{\langle a, s' \rangle : s \rightarrow_S^a s'\}.$$

Notice that the class of all labeled transition systems coincides with the class of all B-coalgebras.

Let us consider now two B-coalgebras $\langle S, \alpha_S \rangle$ and $\langle T, \alpha_T \rangle$. A B-bisimulation between S and T is said to be a relation $R \subseteq S \times T$ satisfying, for all $\langle s, t \rangle \in R$, the following conditions:

- for all s' in S, if $s \rightarrow_S^a s'$, then there is t' in T with $t \rightarrow_T^a t'$ and $\langle s', t' \rangle \in R$,
- for all t' in T, if $t \rightarrow_T^a t'$, then there is s' in S with $s \rightarrow_S^a s'$ and $\langle s', t' \rangle \in R$.

Instead of the induction proof principle used in algebras, the coinduction proof principle is involved into coalgebras: for every bisimulation R on S, $R \subseteq 1_S$ (where $1_S = \{\langle s, s \rangle : s \in S\}$. Equivalently, for all s and s' in S, if s and s' are bisimilar in S, then $s = s'$.

Let M be a labeled transition system $\langle S, \rightarrow_S, A \rangle$. Assume S is a support set for the elementary diagram $Th(\mathsf{S})$ and A consists just of functions of $Th(\mathsf{S})$. To individual constants $c_m, c_n, \ldots$ we can assign meanings in the set S by an interpretation I. In other words, $I(c_m) = m \in S$. We can extend this interpretation I to the interpretation of all functions on S: $I(f(t_1, \ldots, t_n)) = \langle I(t_1), \ldots, I(t_n) \rangle \rightarrow_S^f s' \in S$. In this way by I we assign streams of M to streams of terms of L_M. Evidently, streams $I(f(t_1, \ldots, t_n))$ of M and streams $f(t_1, \ldots, t_n)$ of L_M are bisimilar. Wave sets of M consist of bisimilar streams of wave sets of L_M.

The interpretation I can be extended also to the interpretation of all atomic formulas of L_M as follows. Let R be a relation in M. For each atomic $P \in L_M$, $I(P(t_1, \ldots, t_n)) = R(I(t_1), \ldots, I(t_n))$. For instance, the predicate $f(t_1, \ldots, t_n)) = t_i$ is true if and only if $\langle I(t_1), \ldots, I(t_n) \rangle \rightarrow_S^f I(t_i)$.

Let F be a set of propositions of our formal language L_M. Taking into account that for every predicate P_{ij} there exists a relation R_{ij} of M, we can consider on the basis of M a model M' of F (it is denoted by M'$\models \mathsf{F}$) if the following conditions are satisfied for any formula φ of F and any stream (wave set) of M:

- on the basis of M let us construct a transition system with wave sets, $\mathsf{M}' = \langle S, \rightarrow_S, A \rangle$, where $S = \{0,1\}$ and $A = \{\vee, \wedge, \Rightarrow, \neg\}$, by assuming that for any atomic $P \in L_M$, $I(P) = 1$ in M' if and only if $I(P) = 1$ in M;

• M' |= φ iff the truth valuation I of non-atomic φ is bisimilar with an appropriate stream $I(\varphi)$ of M'.

Let us assume that there is a class K of all coalgebras consisting of the same set of relations. Then there is no proposition φ that is true on any system of the class K. However, there exists a proposition φ that is realizable on the class K, i.e. K contains a system where φ is true.

Let L be a family of propositions. The class of all coalgebraic systems, where all propositions of L are true, is denoted by $Mod(L)$.

Proposition 2. If $L_1 \subseteq L_2$, then $Mod(L_1) \subseteq Mod(L_2)$.

Proof. Let L_1, L_2 be wave sets of formulas such that $L_1 \subseteq L_2$. Assume, $Mod(L_1)$ and $Mod(L_2)$ are wave sets too. Then $Mod(L_1) \subseteq Mod(L_2)$. Q.E.D.

Let K be a class of coalgebraic systems. The family of all propositions that are true on the class K is an elementary theory and is called an elementary theory of the class K. It is denoted by $Th(K)$.

Proposition 3. If $K_1 \subseteq K_2$, then $Th(K_1) \subseteq Th(K_2)$.

V. Conclusion

Propositions 2 – 3 demonstrate some features of unconventional logic: the more propositions, the more models and also the more models, the more propositions.

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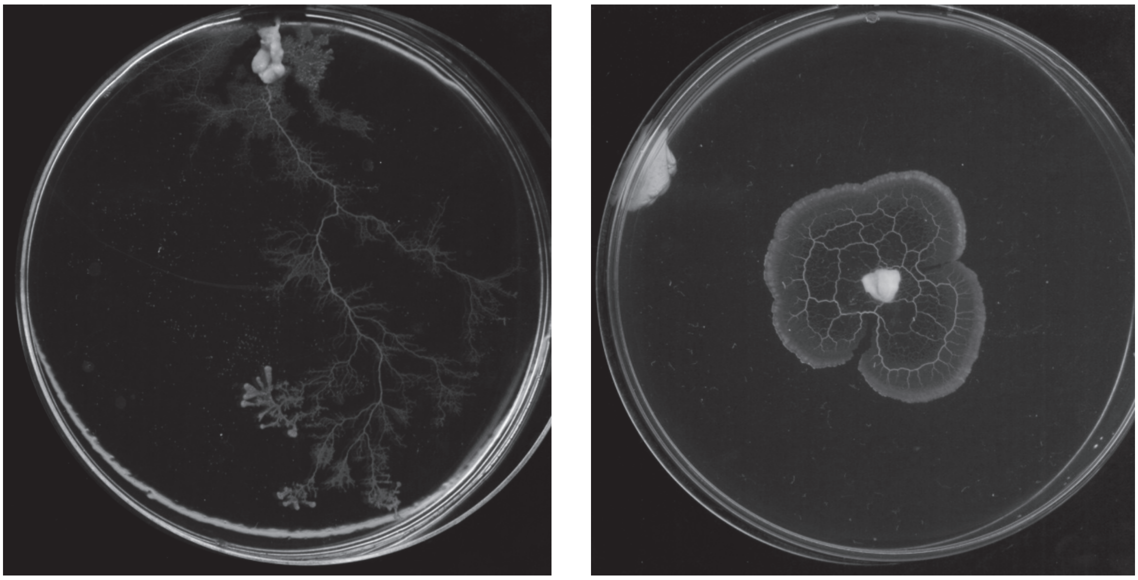

Fig.2. The plasmodium cultivated on a nutrient-rich substrate right and on a nutrient-poor substrate left. *Cortesy of Andy Adamatzky*.

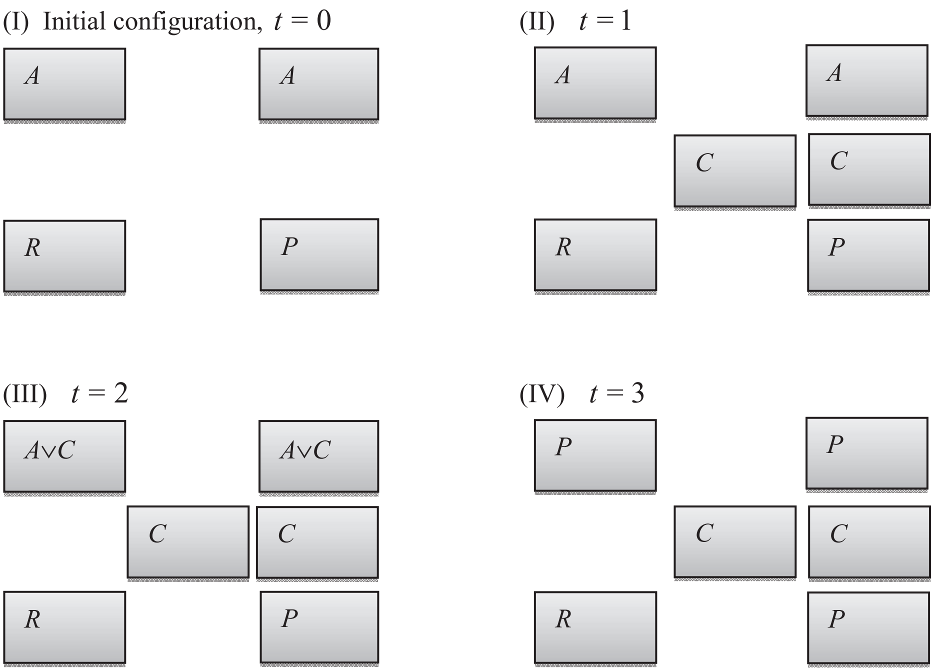

Fig.3. A proof-theoretic spatial automaton with the neighborhood of 8 members in the 2-dimensional space for Physarum polycephalum.